Applied and Computational Harmonic Analysis **11**, 305–312 (2001)
doi:10.1006/acha.2001.0355, available online at http://www.idealibrary.com on IDEAL®

LETTER TO THE EDITOR

Construction of Multivariate Tight Frames via Kronecker Products [1]

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Communicated by Amos Ron
Received November 23, 1999; revised April 19, 2001; published online July 10, 2001

Abstract—Integer-translates of compactly supported univariate refinable functions ϕ_i, such as cardinal B-splines, have been used extensively in computational mathematics. Using certain appropriate direction vectors, the notion of (multivariate) box splines can be generalized to (non-tensor-product) compactly supported multivariate refinable functions Φ from the ϕ_i's. The objective of this paper is to introduce a Kronecker-product approach to build compactly supported tight frames associated with Φ, using the two-scale symbols of the univariate tight frame generators associated with the ϕ_i's. © 2001 Academic Press

1. INTRODUCTION

Extension from univariate function spaces to the multivariate setting can be trivially accomplished by considering the space generated by tensor-products of the univariate basis functions. When these univariate basis functions are compactly supported refinable functions in terms of dilation and translation, not only the procedure for introducing (multivariate) box-splines [1, 3] can be used to create the generating "basis" functions of the multivariate function spaces, but these multivariate generating functions also remain to be refinable with multivariate Laurent polynomial two-scale symbols.

The objective of this paper is to introduce an algorithmic approach for constructing tight frames associated with compactly supported multivariate refinable generating functions. This approach features a recipe for selecting the columns of the Kronecker product of the orthonormal matrix extensions of the univariate two-scale symbols in order to formulate the multivariate orthonormal matrix extension of the multivariate two-scale symbol, which in turn gives the symbols for writing down the desired multivariate tight frame generators. This recipe will be derived in Section 3, where its consequences will be stated as Theorem 1.

[1] Partially supported by NSF Grants DMI-96-34833 and CCR-99-88289, and ARO Grant DAAD-19-00-1-0512.

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Preliminaries on orthonormal matrix extension due to Ron and Shen [6], a criterion for the validity of such extensions observed in our earlier work [4], and a key formula for Kronecker products will be discussed in Section 2. In the final section, we consider the construction of tight frames of box-splines in the three- and four-directional meshes in $\mathbb{R}^2$.

2. PRELIMINARIES

This section is devoted to the introduction of necessary notations and preliminary results.

2.1. *Multivariate Refinable Functions*

Let Φ be a compactly supported refinable function in $L^2(\mathbb{R}^s)$ with two-scale relation

$$\Phi(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^s} p_{\mathbf{k}} \Phi(2\mathbf{x} - \mathbf{k}) \tag{2.1}$$

for some finite sequence $\{p_{\mathbf{k}}\}$, or equivalently,

$$\hat{\Phi}(\underline{\omega}) = P(e^{-i\underline{\omega}/2})\hat{\Phi}(\underline{\omega}/2), \qquad \underline{\omega} \in \mathbb{R}^s,$$

where

$$P(\mathbf{z}) := \frac{1}{2^s} \sum_{\mathbf{k}\in\mathbb{Z}^s} p_{\mathbf{k}} \mathbf{z}^{\mathbf{k}} \tag{2.2}$$

is a Laurent polynomial in $\mathbb{C}^s$. Note that refinable functions are less restrictive than scaling functions (cf. [2, 5]) in that $\{\Phi(\cdot - \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^s\}$ is not required to constitute a Riesz basis of the $L^2(\mathbb{R}^s)$ closure of its algebraic span.

To construct Φ from $t \geq s$ compactly supported univariate refinable functions ϕ_i in $L^2(\mathbb{R})$ with finite two-scale sequences $\{p_{i,k}\}$, or equivalently univariate Laurent polynomials

$$P_k(z) := \frac{1}{2} \sum_{j\in\mathbb{Z}} p_{k,j} z^j, \qquad k = 1, \ldots, t, \tag{2.3}$$

let A be any nonsingular s-dimensional square matrix with row vectors $\mathbf{y}_1, \ldots, \mathbf{y}_s$, and introduce additional vectors

$$\mathbf{y}_v = \sum_{k=1}^{s} \eta_{v,k} \mathbf{y}_k, \qquad \eta_{v,k} \in \mathbb{Z}, \tag{2.4}$$

$s < v \leq t$. Then Φ, defined by its Fourier transform,

$$\hat{\Phi}(\underline{\omega}) := \prod_{i=1}^{t} \hat{\phi}_i(\underline{\omega} \cdot \mathbf{y}_i), \qquad \underline{\omega} \in \mathbb{R}^s, \tag{2.5}$$

is a refinable function with two-scale s-variate Laurent polynomial symbol

$$P(\mathbf{z}) = P(z_1, \ldots, z_s) := \prod_{k=1}^{s} P_k(z_k) \prod_{v=s+1}^{t} P_v\left(\prod_{\ell=1}^{s} z_\ell^{\eta_{v,\ell}}\right). \tag{2.6}$$

Here, $z_k := e^{-i\underline{\omega}\cdot\mathbf{y}_k/2}$, $k = 1, \ldots, s$, and for $s < v \le t$,

$$z_v = e^{-i\sum_{k=1}^{s}\eta_{v,k}\underline{\omega}\cdot\mathbf{y}_k/2} = \prod_{k=1}^{s} z_k^{\eta_{v,k}}. \tag{2.7}$$

Indeed, by using (2.5) and (2.6), we have

$$\begin{aligned}\hat{\Phi}(\underline{\omega}) &= \prod_{k=1}^{t}\hat{\phi}_k(\underline{\omega}\cdot\mathbf{y}_k) = \prod_{k=1}^{t} P_k(e^{-i\underline{\omega}\cdot\mathbf{y}_k/2})\hat{\phi}_k(\underline{\omega}\cdot\mathbf{y}_k/2)\\ &= P(e^{-i\underline{\omega}/2})\hat{\Phi}(\underline{\omega}/2).\end{aligned}$$

2.2. *Orthonormal Matrix Extensions*

Let $P(\mathbf{z})$ be an s-variate Laurent polynomial. For every $\ell \epsilon \{0, \ldots, 2^s - 1\}$, consider its (unique) binary representation

$$\ell = \sum_{k=1}^{s}\epsilon_k^{\ell}2^{k-1}, \qquad \epsilon_k^{\ell} \in \{0, 1\}, \tag{2.8}$$

or equivalently,

$$\ell \mapsto \sigma(\ell) := (\epsilon_1^{\ell}, \ldots, \epsilon_s^{\ell}), \tag{2.9}$$

and adopt the standard multivariate notation

$$(-1)^{\sigma(\ell)}\mathbf{z} = ((-1)^{\epsilon_1^{\ell}}z_1, \ldots, (-1)^{\epsilon_s^{\ell}}z_s), \tag{2.10}$$

where $\mathbf{z} := (z_1, \ldots, z_s)$. The problem of orthonormal matrix extension is to find s-variate Laurent polynomials $Q_1(\mathbf{z}), \ldots, Q_N(\mathbf{z})$, such that the matrix

$$R_N(\mathbf{z}) := \begin{bmatrix} P((-1)^{\sigma(0)}\mathbf{z}) & \cdots & P((-1)^{\sigma(2^s-1)}\mathbf{z}) \\ Q_1((-1)^{\sigma(0)}\mathbf{z}) & \cdots & Q_1((-1)^{\sigma(2^s-1)}\mathbf{z}) \\ \vdots & \ddots & \vdots \\ Q_N((-1)^{\sigma(0)}\mathbf{z}) & \cdots & Q_N((-1)^{\sigma(2^s-1)}\mathbf{z}) \end{bmatrix} \tag{2.11}$$

satisfies

$$R_N^*(\mathbf{z})R_N(\mathbf{z}) = I_{2^s}, \qquad \mathbf{z} \in T^s. \tag{2.12}$$

Here, T denotes the unit circle in $\mathbb{C}$ and the subscript N, which denotes the number of Laurent polynomials $Q_i(\mathbf{z})$ to be constructed, is called the order of the matrix extension $R_N(\mathbf{z})$. The importance of this orthonormal matrix extension problem is that under very mild conditions on the refinable function Φ with two-scale Laurent symbol $P(\mathbf{z})$, Ron and Shen [6] proved that the compactly supported s-variate functions Ψ_ℓ, $\ell = 1, \ldots, N$, with

$$\hat{\Psi}_\ell(\underline{\omega}) = Q_\ell(e^{-i\underline{\omega}/2})\hat{\Phi}(\underline{\omega}/2), \qquad \ell = 1, \ldots, N, \tag{2.13}$$

generate a tight frame

$$\{2^{sj/2}\Psi_\ell(2^j\mathbf{x}-\mathbf{k}) : j \in \mathbb{Z},\ \mathbf{k} \in \mathbb{Z}^s,\ \text{and } \ell = 1, \ldots, N\} \tag{2.14}$$

of $L^2(\mathbb{R}^s)$.

For the univariate setting, it was shown in our earlier work [4] that a Laurent polynomial $P(z)$ admits orthonormal matrix extension, if and only if it satisfies

$$|P(z)|^2 + |P(-z)|^2 \leq 1, \qquad z \in T. \tag{2.15}$$

Moreover, if (2.15) holds, two procedures were given in [4], one for constructing orthonormal matrix extension of order 2, and the other for constructing symmetric/antisymmetric orthonormal matrix extension of order 3 for symmetric $P(z)$.

2.3. Kronecker Products of Rectangular Matrices

The objective of this paper is to introduce a Kronecker-product approach to solving the orthonormal matrix extension problem (2.12) for constructing tight frames of $L^2(\mathbb{R}^s)$. Recall that the Kronecker product of an $m \times n$ matrix $A = [a_{ij}]$ with a $p \times q$ matrix $B = [b_{ij}]$ is an $mp \times nq$ matrix, defined by

$$A \otimes B = [a_{ij}B], \tag{2.16}$$

where block matrix formulation is used. As an immediate consequence of the property

$$(A \otimes B)^*(A \otimes B) = (A^*A) \otimes (B^*B) \tag{2.17}$$

of Kronecker products for rectangular matrices of arbitrary dimensions, where the asterisk indicates a complex conjugate of the transpose matrix, we have the following.

LEMMA 1. *Let A and B be rectangular matrices of dimensions $m \times n$ and $p \times q$ with $m \geq n$ and $p \geq q$, respectively, such that $A^*A = I_n$ and $B^*B = I_q$. Then*

$$(A \otimes B)^*(A \otimes B) = I_{nq}. \tag{2.18}$$

3. COMPACTLY SUPPORTED MULTIVARIATE TIGHT FRAMES

Let $t \geq s$ and $P_1(z), \ldots, P_t(z)$ be univariate two-scale symbols as in (2.3) that admit orthonormal matrix extensions $R_{N_1}(z), \ldots, R_{N_t}(z)$ of orders $N_1, \ldots, N_t$, respectively. By using the notation in (2.7) and observing that the Kronecker product operation is associative, we introduce the $(N+1) \times 2^t$ matrix

$$K_N(\mathbf{z}) = R_{N_1}(z_1) \otimes \cdots \otimes R_{N_t}(z_t), \tag{3.1}$$

where $\mathbf{z} = (z_1, \ldots, z_s)$ is extended from T^s to $\mathbb{C}^s$, and

$$N := (N_1 + 1) \cdots (N_t + 1) - 1. \tag{3.2}$$

Since each $P_i(z)$ admits an orthonormal matrix extension $R_{N_i}(z)$ of order N_i, $i = 1, \ldots, t$, it follows from Lemma 1 that

$$K_N^*(\mathbf{z})K_N(\mathbf{z}) = I_{2^t}, \qquad \mathbf{z} \in T^s. \tag{3.3}$$

An important property of the matrix $K_N(\mathbf{z})$ is that its first row contains all of the entries of the first row of the matrix $R_N(\mathbf{z})$ in (2.11) for the mapping $\sigma(\ell)$ defined in (2.9). To see this, recall that for each $\ell = 0, \ldots, 2^s - 1$, $\sigma(\ell) \in \{0, 1\}^s$ is uniquely determined by the binary representation of ℓ in (2.8), so that by introducing the notation

$$\epsilon_v^\ell := \left|\sum_{k=1}^{s} \epsilon_k^\ell \eta_{v,k}\right| \pmod 2, \qquad v = s+1, \ldots, t, \tag{3.4}$$

we have

$$P((-1)^{\sigma(\ell)}\mathbf{z}) = \prod_{i=1}^{t} P_i((-1)^{\epsilon_i^\ell} z_i), \qquad \ell = 0, \ldots, 2^s - 1, \tag{3.5}$$

where the notation in (2.7) is used. But from (2.6) and the definition of $K_N(\mathbf{z})$, we see that the 2^t components of the first row vector of $K_N(\mathbf{z})$ are given by $P_1(\pm z_1) \cdots P_t(\pm z_t)$ with all possible 2^t combinations of $\pm$ signs, so that $P((-1)^{\sigma(\ell)}\mathbf{z})$ in (3.5) is one of these entries. To be more precise, we observe, by (2.16) and (3.1), that the first row block matrices of $K_N(\mathbf{z})$ have the form

$$[P_1(z_1)R_{N_2}(z_2) \otimes \cdots \otimes R_{N_t}(z_t),\ P_1(-z_1)R_{N_2}(z_2) \otimes \cdots \otimes R_{N_t}(z_t)]. \tag{3.6}$$

Consequently, for each $\ell = 0, \ldots, 2^s - 1$, the $(\epsilon_1^\ell 2^{t-1} + \epsilon_2^\ell 2^{t-2} + \cdots + \epsilon_t^\ell 2^0 + 1)$th entry of the first row of (3.6) is $P((-1)^{\sigma(\ell)}\mathbf{z})$ for $\ell = 0, \ldots, 2^s - 1$. Thus, by using the notation

$$d_\ell := \sum_{i=1}^{t} \epsilon_i^\ell 2^{t-i}, \qquad \ell = 0, \ldots, 2^s - 1, \tag{3.7}$$

we may construct an $(N+1) \times 2^s$ matrix $R_N(\mathbf{z})$, with exactly the same formulation as (2.11) by using the $(d_\ell + 1)$th column of $K_N(\mathbf{z})$ as the $(\ell + 1)$th column of $R_N(\mathbf{z})$, $\ell = 0, \ldots, 2^s - 1$. By (3.3), we see that this $R_N(\mathbf{z})$ satisfies (2.12).

As a consequence of the above discussion, and by applying the results stated in Section 2.2, we have established the following result.

THEOREM 1. *Let $t \geq s$ and $P_1(z), \ldots, P_t(z)$ be univariate Laurent polynomials that admit orthonormal matrix extensions of orders $N_1, \ldots, N_t$, respectively. Then the s-variate Laurent polynomial $P(\mathbf{z})$, defined in* (2.6) *by using* (2.4)*, also admits an orthonormal matrix extension $R_N(\mathbf{z})$ given by* (2.11) *with N given by* (3.2)*, for some s-variate Laurent polynomials $Q_\ell(\mathbf{z})$, $\ell = 1, \ldots, N$.*

Furthermore, if $\phi_1, \ldots, \phi_t$ are univariate refinable functions with two-scale symbols $P_1(z), \ldots, P_t(z)$, respectively, such that $\hat{\phi}_1(0), \ldots, \hat{\phi}_t(0) \neq 0$ and $P_1(-1) = \cdots = P_t(-1) = 0$, then the Laurent polynomials $Q_\ell(\mathbf{z})$ are the two-scale symbols of the compactly supported $L^2(\mathbb{R}^s)$ functions Ψ_ℓ, $\ell = 1, \ldots, N$, in (2.13)*, that generate a tight frame* (2.14) *of $L^2(\mathbb{R}^s)$.*

We remark that if the univariate Laurent polynomials in the above theorem are symmetric, then symmetric or antisymmetric frame generators $\Psi_1, \ldots, \Psi_N$ can be constructed, in view of the fact that the Kronecker product operation preserves symmetry. In addition, since the largest value of $N_1, \ldots, N_t$ is 2, the number of frame generators Ψ_ℓ's is between $2^t - 1$ and $3^t - 1$; but if symmetry or antisymmetry is desired, this number of frame generators may increase up to $4^t - 1$. The reason for the lower bound is that if ϕ_ℓ is an orthonormal scaling function in $L^2(\mathbb{R})$, then we may choose $N_\ell = 1$.

4. BOX-SPLINE TIGHT FRAMES

Let $M_n(x)$ denote the cardinal B-spline of order n [3, pp. 1 and 2]. We choose $\mathbf{y}_1 = \mathbf{e}_1$, $\mathbf{y}_2 = \mathbf{e}_2$, $\mathbf{y}_3 = \mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{y}_4 = \mathbf{e}_1 - \mathbf{e}_2$ in (2.4), and $\phi_1 := M_l$, $\phi_2 := M_m$, $\phi_3 := M_n$, $\phi_4 := M_p$ in (2.5) to define $\hat{\Phi}$. For $\ell, m, n, p > 0$, this bivariate refinable function Φ is the box-spline $M_{\ell mnp}$ on a four-directional mesh with Fourier transform given by

$$\begin{aligned}
\hat{\Phi}(\underline{\omega}) &= \hat{M}_{\ell mnp}(\underline{\omega}) = \hat{M}_\ell(\underline{\omega} \cdot \mathbf{e}_1)\hat{M}_m(\underline{\omega} \cdot \mathbf{e}_2)\hat{M}_n(\underline{\omega} \cdot (\mathbf{e}_1 + \mathbf{e}_2))\hat{M}_p(\underline{\omega} \cdot (\mathbf{e}_1 - \mathbf{e}_2)) \\
&= \hat{M}_\ell(\omega_1)\hat{M}_m(\omega_2)\hat{M}_n(\omega_1 + \omega_2)\hat{M}_p(\omega_1 - \omega_2) \\
&= \left(\frac{1 - e^{-i\omega_1}}{i\omega_1}\right)^\ell \left(\frac{1 - e^{-i\omega_2}}{i\omega_2}\right)^m \left(\frac{1 - e^{-i(\omega_1+\omega_2)}}{i(\omega_1 + \omega_2)}\right)^n \left(\frac{1 - e^{-i(\omega_1-\omega_2)}}{i(\omega_1 - \omega_2)}\right)^p,
\end{aligned}$$

and two-scale symbol

$$p(\mathbf{z}) = P(z_1, z_2) = \left(\frac{1 + z_1}{2}\right)^\ell \left(\frac{1 + z_2}{2}\right)^m \left(\frac{1 + z_1 z_2}{2}\right)^n \left(\frac{1 + z_1/z_2}{2}\right)^p.$$

For details, see [1] and [3, pp. 15–18]. We remark that the $\mathbb{Z}^2$-translates of $M_{\ell mnp}$ do not constitute a Riesz basis of the $L^2(\mathbb{R}^2)$-closure of their algebraic span.

For $p = 0$, $\Phi = M_{\ell mn0} =: M_{\ell mn}$ are box-splines on a three-directional mesh. The simplest such example is the Courant element M_{111}, whose two-scale symbol has a seventh-order orthonormal matrix extension $R_7(\mathbf{z})$, formulated via the Kronecker product

$$K_7(\mathbf{z}) = K_7(z_1, z_2) = \begin{bmatrix} \frac{1+z_1}{2} & \frac{1-z_1}{2} \\ \frac{1-z_1}{2} & \frac{1+z_1}{2} \end{bmatrix} \otimes \begin{bmatrix} \frac{1+z_2}{2} & \frac{1-z_2}{2} \\ \frac{1-z_2}{2} & \frac{1+z_2}{2} \end{bmatrix} \otimes \begin{bmatrix} \frac{1+z_1 z_2}{2} & \frac{1-z_1 z_2}{2} \\ \frac{1-z_1 z_2}{2} & \frac{1+z_1 z_2}{2} \end{bmatrix}.$$

Recall that the nth column of $R_7(\mathbf{z})$ is the $(d_{n-1} + 1)$th column of $K_7(\mathbf{z})$, for $n = 1, \ldots, 4$, where d_ℓ is given by (3.7). That is, $R_7(\mathbf{z})$ is an 8×4 matrix whose column vectors are given, in consecutive order, by the first, sixth, fourth, and seventh columns of $K_7(\mathbf{z})$. From this, we can write the two-scale symbols of the tight frame generators $\Psi_1, \ldots, \Psi_7$, as follows.

$$\begin{aligned}
Q_1(\mathbf{z}) &= \left(\frac{1 + z_1}{2}\right)\left(\frac{1 + z_2}{2}\right)\left(\frac{1 - z_1 z_2}{2}\right), \\
Q_2(\mathbf{z}) &= \left(\frac{1 + z_1}{2}\right)\left(\frac{1 - z_2}{2}\right)\left(\frac{1 + z_1 z_2}{2}\right), \\
Q_3(\mathbf{z}) &= \left(\frac{1 + z_1}{2}\right)\left(\frac{1 - z_2}{2}\right)\left(\frac{1 - z_1 z_2}{2}\right),
\end{aligned}$$

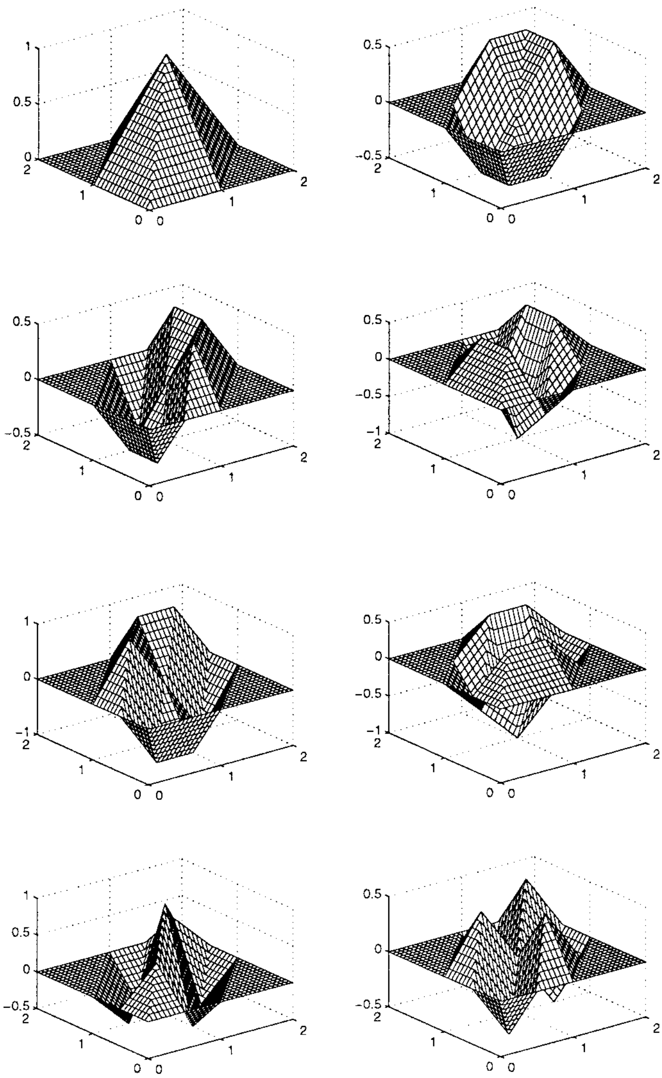

FIG. 1. Box-spline M_{111} and tight frame generators $\Psi_1, \ldots, \Psi_7$.

$$Q_4(\mathbf{z}) = \left(\frac{1-z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{1+z_1z_2}{2}\right),$$

$$Q_5(\mathbf{z}) = \left(\frac{1-z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{1-z_1z_2}{2}\right),$$

$$Q_6(\mathbf{z}) = \left(\frac{1-z_1}{2}\right)\left(\frac{1-z_2}{2}\right)\left(\frac{1+z_1z_2}{2}\right),$$

$$Q_7(\mathbf{z}) = \left(\frac{1-z_1}{2}\right)\left(\frac{1-z_2}{2}\right)\left(\frac{1-z_1z_2}{2}\right).$$

Observe that $\Psi_1, \ldots, \Psi_7$ are either symmetric or antisymmetric. (See the pictures of these functions along with $\Phi = M_{111}$ in Fig. 1.)

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